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On global $SL(2,\mathbb{R})$ symmetries of differential operators

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Abstract

This paper studies the Lie symmetries of the equation

$$\left[\partial_x^2 + ax^{-1}\partial_x + b\partial_t\right] f(x,t) = 0.$$

Generically the symmetry group is $\mathfrak{sl}(2,\mathbb{R})$. In particular, we show the local action of the symmetry group extends to a global representation of $SL(2,\mathbb{R})$ on an appropriate subspace of smooth solutions. In fact, every principal series is realized in this way. Moreover, this subspace is naturally described in terms of sections of an appropriate line bundle on which the given differential operator is intimately related to the Casimir element.

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1. Introduction

The origin of Lie theory can be traced back to Sophus Lie's development of his so-called prolongation algorithm for calculating symmetries of partial differential equations (e.g., [7]). Though tedious, the method is completely elementary and often produces very interesting answers. This mainly stems from the fact that though these symmetries preserve the solution space of a given differential operator, they typically do not preserve the differential operator itself.

For example, the symmetries of the heat equation $u_{xx} = u_t$ or the Schrödinger equation $u_{xx} = iu_t$ are well known. Modulo an infinite-dimensional piece, the symmetries turn out to be isomorphic to $\mathfrak{sl}(2, \mathbb{R}) \ltimes \mathfrak{h}_3(\mathbb{R})$ where $\mathfrak{h}_3(\mathbb{R})$ is the three-dimensional Heisenberg Lie algebra. Moreover, the action exponentiates *locally* to an action of the corresponding Lie group on a neighborhood of the identity—though not usually to the whole group. For instance, it is easy to check that the strictly lower triangular matrices locally act on solutions to the heat equation by

$$\left(\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} u\right)(x, t) = \frac{1}{\sqrt{1-ct}} e^{\frac{cx^2}{4(1-ct)}} u\left(\frac{x}{1-ct}, \frac{t}{1-ct}\right)$$

whenever $ct < 1$. This action is far from obvious. As phrased in this language, the fact that, say, the heat equation even has any type of $\mathfrak{sl}(2, \mathbb{R})$ symmetry is rather surprising.

Though Lie's theory of prolongation is a very simple algorithm, on the whole it has been little used in representation theory. This is due to the enormous difficulty posed by only having a local action of a Lie group instead of a global representation. However, in [2,3], Craddock made an important discovery. He found that in certain cases a global representation was achieved by restricting to an appropriate subset of the solution space. For instance in the case of the heat equation, Craddock showed there exists a special subset of distributional solutions to the heat equation on which the local action of $SL(2, \mathbb{R}) \ltimes H_3(\mathbb{R})$ given by the prolongation algorithm actually extends to a global representation. In fact, it turns out to be irreducible nonunitary under $SL(2, \mathbb{R})$. However even in this case, there still seems to be no reason why the heat equation ought to have, say, $SL(2, \mathbb{R})$ appearing in its symmetry group. In Craddock's treatment, the special subset of solutions upon which the local action extends to a global action is mysterious. It is only described as the image under the analytic continuation of a certain integral transform of the span of certain types of exponential functions. No natural description is given.

Motivated by the Heat and Schrödinger equations, in this paper we study the family of possibly singular differential equations

$$\left[\partial_x^2 + ax^{-1}\partial_x + b\partial_t\right] f(x, t) = 0 \tag{1.1}$$

for $a, b \in \mathbb{C}$. In particular, for $a = 0$ and $b = -1, -i$, we recover the Heat and Schrödinger equations, respectively. Modulo an infinite-dimensional piece (reflecting the fact that the solution space is a vector space), the symmetry group is generically

$\mathfrak{sl}(2, \mathbb{R})$. We prove the local action of the symmetry group on a certain subspace of solutions actually extends to a global representation of $SL(2, \mathbb{R})$, i.e., the Lie algebra representation exponentiates to an action of the entire Lie group instead of simply a neighborhood of the identity. Moreover, we give a very natural and simple description of this special subspace of solutions admitting a global action. Namely, this special space of functions arises from sections of a certain line bundle restricted to an open dense copy of $\mathbb{R}^2$. Additionally, there is a simple explanation of why these differential operators admit $\mathfrak{sl}(2, \mathbb{R})$ symmetry. Remarkably, we show these differential operators in Eq. (1.1) are very nearly the Casimir element acting on smooth sections of the appropriate line bundle. In fact, when viewed as an operator between two different line bundles, these differential operators actually become intertwining maps. Finally, it turns out that the set of representations appearing as smooth solutions to Eq. (1.1) encompasses a very rich family of representations of $SL(2, \mathbb{R})$. In fact, every principal series representation of $SL(2, \mathbb{R})$ appears. We note that in the special case of the Heat or Schrödinger equations, respectively, we construct two different representations of $SL(2, \mathbb{R})$, none of which are isomorphic to the one given in [2] (see Theorem 8 for exact parameters).

The paper is organized as follows. In Section 2, we discuss the symmetry group for Eq. (1.1) via Lie's prolongation method. In Section 3, we introduce the necessary line bundles. Motived by the Heat and Schrödinger equations, this means studying induced representations of $G = SL(2, \mathbb{R}) \ltimes H_3(\mathbb{R})$. Here the Casimir element quickly gives rise to the differential operator in Eq. (1.1). In Section 4, we prove general intertwining theorems and deal with the case of $b = 0$. In Section 5, we work with the general case of $b \neq 0$. In both sections, we identify the representations under consideration up to infinitesimal equivalence. Finally in Section 6 we address matters of norms and explicit intertwining operators.

2. Symmetries

Following standard techniques in Lie's prolongation algorithm (e.g., [7]), it is straightforward to calculate the symmetry group of the partial differential equation

$$\left(\nabla_{r,s} f\right)(x, t) = 0, \tag{2.1}$$

where

$$\nabla_{r,s} = \partial_x^2 - (1 + 2r)x^{-1}\partial_x + 4s\partial_t$$

for $r, s \in \mathbb{C}$. In this case, the procedure yields a Lie algebra of first-order differential operators that exponentiate to a local action of a Lie group on solutions to Eq. (2.1). Since Eq. (2.1) is linear, there is an infinite-dimensional set of symmetries arising by adding one solution to another. In the theorem below, we calculate a complement to this infinite-dimensional set and will write $\mathrm{Sym}_{r,s}$ for this set of remaining symmetries.

We also use the notation $\mathfrak{h}_3(\mathbb{R}) = \mathrm{Lie}(H_3(\mathbb{R}))$. As the details are elementary and not to the heart of this paper, they are omitted.

Theorem 1. *Let $s \neq 0$. Modulo the infinite-dimensional set of symmetries arising by adding one solution to another, the set of symmetries of $\nabla_{r,s} f = 0$, $\mathrm{Sym}_{r,s}$, is generically isomorphic to $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R}$. The central element of $\mathrm{Sym}_{r,s}$ corresponds to multiplying solutions by a constant. The $\mathfrak{sl}(2,\mathbb{R})$ component of $\mathrm{Sym}_{r,s}$ is spanned by the following standard triple*:

$$h = -x\partial_x - 2t\partial_t + r,$$

$$e^+ = -\partial_t,$$

$$e^- = xt\partial_x + t^2\partial_t - (rt + sx^2).$$

Additional symmetries arise in exactly the following two cases.

(1) *If $r = -\frac{1}{2}$, then $\mathrm{Sym}_{r,s} \cong \mathfrak{sl}(2,\mathbb{R}) \ltimes \mathfrak{h}_3(\mathbb{R})$. The $\mathfrak{h}_3(\mathbb{R})$ component is spanned by*

$$v_1 = -\partial_x,$$

$$v_2 = t\partial_x - 2sx,$$

$$\omega = s.$$

(2) *If $r = -\frac{3}{2}$, then $\mathrm{Sym}_{r,s} \cong \mathfrak{sl}(2,\mathbb{R}) \ltimes \mathfrak{h}_3(\mathbb{R})$ The $\mathfrak{h}_3(\mathbb{R})$ component is spanned by*

$$v_1 = -\partial_x - x^{-1},$$

$$v_2 = t\partial_x - 2sx + x^{-1}t,$$

$$\omega = s.$$

We will see that Theorem 7 precisely captures a global action for the symmetry group in all cases except $r = -\frac{3}{2}$. In the case of $r = -\frac{3}{2}$, we will see $H_3(\mathbb{R})$ does not act via its standard action coming from a certain induced representation. However, Theorem 1 does indicate the possibility of some *other* action of $H_3(\mathbb{R})$ occurring on $\ker \nabla_{-\frac{3}{2},s}$. Exponentiating $\partial_x + x^{-1}$ in this case leads to an action of $y \in \mathbb{R}$ by $(y \cdot f)(x,t) = \frac{x+y}{x} f(x+y,t)$. Similarly, exponentiating $t\partial_x - 2sx + x^{-1}t$ leads to an action $(y \cdot f)(x,t) = \frac{x+yt}{x} e^{2ysx - y^2 st} f(x+yt,t)$. Though these actions map solutions to solutions, they do not preserve smooth functions and so will not be studied here.

3. Line bundles

Let

$$G = SL(2,\mathbb{R}) \ltimes H_3(\mathbb{R}),$$

where $H_3(\mathbb{R})$ is the three-dimensional real Heisenberg group. As usual, identify $H_3(\mathbb{R})$ as $\mathbb{R}^2 \times \mathbb{R}$ with group structure $(v,w)(v',w') = (v+v', \langle v,v'\rangle + w + w')$ for any $v, v' \in \mathbb{R}^2$ and $w, w' \in \mathbb{R}$. Here $\langle\cdot,\cdot\rangle$ is the symplectic form given by $\langle v, v'\rangle = v^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} v'$. The action of $SL(2,\mathbb{R})$ on $H_3(\mathbb{R})$ in the semidirect product arises from the natural action of $SL(2,\mathbb{R}) = Sp(2,\mathbb{R})$ on $\mathbb{R}^2$. In particular, for $(A,(v,w)), (A',(v',w')) \in G$, the group structure is given by $(A,(v,w))(A',(v',w')) = (AA', (A'^{-1}v + v', \langle v, A'v'\rangle + w + w'))$.

Let $\overline{B}$ be the subgroup of lower triangular matrices in $SL(2,\mathbb{R})$, $\overline{V}$ be the subgroup of $H_3(\mathbb{R})$ consisting of $\{((0,v_2),w) \mid v_2, w \in \mathbb{R}\}$, and $\overline{P} = \overline{B} \ltimes \overline{V}$ be the corresponding subgroup of G. The set of characters of $\overline{P}$ may be indexed by triples (n,r,s) with $n \in \mathbb{Z}$ (determined only up to parity) and $r, s \in \mathbb{C}$ by defining

$$\chi_{n,r,s}\left(\begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix}, ((0,v_2),w)\right) = \operatorname{sgn}^n(a)\, |a|^r\, e^{sw}.$$

Clearly only the parity of n is relevant for the definition of $\chi_{n,r,s}$, but we adopt this convention over the traditional $\pm$ in order to simplify later notation and avoid ambiguity when a second $\pm$ is introduced.

The smooth unnormalized induced representations we study are

$$\begin{aligned} I(n,r,s) &= \operatorname{Ind}_{\overline{P}}^{G}(\chi_{n,r,s}) \\ &= \{\phi : G \to \mathbb{C}, \phi \in C^\infty \mid \phi(g\overline{p}) = \chi_{n,r,s}^{-1}(\overline{p})\phi(g), \text{ any } g \in G,\ \overline{p} \in \overline{P}\} \end{aligned}$$

with action $(g\phi)(g') = \phi(g^{-1}g')$ for $g, g' \in G$ and $\phi \in I(n,r,s)$.

As $I(n,r,s)$ may be viewed as the smooth sections of the line bundle $G/\overline{P} \times_{\chi_{n,r,s}} \mathbb{C}$ (e.g., [8]), it is possible to restrict to an appropriate open dense set of $G/\overline{P}$ on which the line bundle trivializes and work with functions on this set. In this case, let $N = \{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, ((x,0),0)\}$, a subgroup of G. Identify N with $\mathbb{R}^2$ by $(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, ((x,0),0)) \longleftrightarrow (x,t)$ and let $\pi : \mathbb{R}^2 \cong N \to G/\overline{P}$ be the natural projection. The map π is injective and has open dense range. Thus the *restriction map* sending $\phi \to \phi \circ \pi$ for $\phi \in I(n,r,s)$ is an injective map from $I(n,r,s)$ to $C^\infty(\mathbb{R}^2)$. Write $I'(n,r,s)$ for the image of $I(n,r,s)$ under this restriction and use the restriction map to endow $I'(n,r,s)$ with a G action so that $I'(n,r,s) \cong I(n,r,s)$ as G-modules. Note $I'(n,r,s) \subsetneq C^\infty(\mathbb{R}^2)$ and we reserve the notation $f(x,t)$ for denoting an element of $I'(n,r,s)$ is these coordinates. If G were semisimple, this would simply be the standard noncompact picture ([5, §VII]) of

$I(n,r,s)$. Theorem 2 below is straightforward to verify and follows fairly rapidly from the equation

$$\begin{aligned}
&\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, ((v_1, v_2), w)\right]\left[\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, ((x,0),0)\right] \\
&= \left[\begin{pmatrix} 1 & \dfrac{at+b}{ct+d} \\ 0 & 1 \end{pmatrix}, \left(\left(\frac{v_1 - tv_2 + x}{ct+d}, 0\right), 0\right)\right] \\
&\quad \times \left[\begin{pmatrix} \dfrac{1}{ct+d} & 0 \\ c & ct+d \end{pmatrix}, \left(\left(0, \frac{c(v_1 - tv_2 + x)}{ct+d} + v_2\right),\right.\right. \\
&\quad \left.\left. \times \left(\frac{-c(v_1 - tv_2 + x)}{ct+d} - v_2\right)(v_1 - tv_2 + x) - v_2 x + w\right)\right].
\end{aligned}$$

As the calculations are both elementary and similar to typical induction calculations, details are omitted.

Theorem 2. *Let $f \in I'(n,r,s)$. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$,*

$$(gf)(x,t) = \operatorname{sgn}^n(-ct+a)\,|-ct+a|^r\, e^{-\frac{scx^2}{-ct+a}}\, f\left(\frac{x}{-ct+a}, \frac{dt-b}{-ct+a}\right). \tag{3.1}$$

For $h = ((v_1, v_2), w) \in H_3(\mathbb{R})$,

$$(hf)(x,t) = e^{s[v_1 v_2 - t v_2^2 - 2x v_2 + w]}\, f(x - v_1 + t v_2, t).$$

The element $\begin{pmatrix} \alpha & \beta \\ \gamma & \alpha \end{pmatrix}$ in the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ acts on $I'(n,r,s)$ by the first-order differential operator

$$(-\alpha x + \gamma x t)\partial_x + (-2\alpha t - \beta + \gamma t^2)\partial_t + (\alpha r - \gamma r t - \gamma s x^2)$$

and the Casimir element for $\mathfrak{sl}(2,\mathbb{R})$, in standard notation $\Omega = 2e^- e^+ + h + \frac{1}{2}h^2$, acts by

$$\frac{1}{2}x^2 \nabla_{n,r,s} + \frac{1}{2} r(r+2) \ \textit{where}$$

$$\nabla_{n,r,s} \equiv \partial_x^2 - (1+2r)x^{-1}\partial_x + 4s\partial_t.$$

Finally, the element $((v_1, v_2), \omega)$ *in* $\mathrm{Lie}(H_3(\mathbb{R}))$ *acts by the first-order differential operator*

$$(-v_1 + v_2 t)\partial_x + (-2sv_2 x + \omega s).$$

Note there is no dependence on n in the above formula for $\nabla_{n,r,s}$. Its presence in the symbol persists to help specify its domain, $I'(n,r,s)$.

Corollary 1. *The action of* $SL(2,\mathbb{R})$ (*or* G *when* $r = -\frac{1}{2}$) *on* $I'(n,r,s)$ *is a global extension of the local action of* $SL(2,\mathbb{R})$ (*or* G *when* $r = -\frac{1}{2}$) *given by Lie's prolongation method applied to the operator* $\nabla_{r,s} f = 0$.

It is also useful to realize $I(n,r,s)$ another way. Write $K = SO(2,\mathbb{R})$ for the maximal compact subgroup of $SL(2,\mathbb{R})$. Then $G/\overline{P} \cong K/\{\pm I\} \times \{((y,0),0)\}_{y\in\mathbb{R}}$ as manifolds and so $I(n,r,s) \cong C^\infty(K/\{\pm I\} \times \{((y,0),0)\}_{y\in\mathbb{R}})$ as vector spaces. Writing $K_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, we can identify the vector space $I(n,r,s)$ with

$$I''(n,r,s) = \{F(y,\theta) \in C^\infty(\mathbb{R}^2) \mid F(y,\theta+\pi) = (-1)^n F(-y,\theta)\}$$

by mapping $\phi \to F$ where $F(y,\theta) = \phi((K_\theta, ((y,0),0))\overline{P})$ for $\phi \in I(n,r,s)$. Observe the vector space for $I''(n,r,s)$ is independent of r and s. We use the above map to carry over the G action so $I(n,r,s) \cong I''(n,r,s)$ as G-modules and we reserve the notation $F(y,\theta)$ for denoting an element of $I''(n,r,s)$ is these coordinates. If G were semisimple, this would simply be the usual compact picture [5, §VII]. Theorem 3 below is straightforward to verify and follows fairly rapidly from the equation

$$\begin{aligned} &\left[\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, ((x,0),0)\right] \\ &\quad = \left[K_{\arctan t}, \left(\left(\frac{x}{\sqrt{1+t^2}}, 0\right), 0\right)\right] \\ &\quad\quad \times \left[\begin{pmatrix} \frac{1}{\sqrt{1+t^2}} & 0 \\ \frac{t}{\sqrt{1+t^2}} & \sqrt{1+t^2} \end{pmatrix}, \left(\left(0, \frac{xt}{1+t^2}\right), \frac{-tx^2}{1+t^2}\right)\right]. \end{aligned}$$

Again, as the calculations are both elementary and similar to typical induction calculations, details are omitted.

Theorem 3. *Under the G-isomorphism* $I'(n,r,s) \cong I''(n,r,s)$, $f \in I'(n,r,s)$ *corresponds to* $F \in I''(n,r,s)$ *if and only if*

$$f(x,t) = (1+t^2)^{\frac{r}{2}} e^{\frac{stx^2}{1+t^2}} F\left(\frac{x}{\sqrt{1+t^2}}, \arctan t\right) \tag{3.2}$$

and we write the isomorphism $\psi_{n,r,s} : I'(n,r,s) \to I''(n,r,s)$ that is given by

$$\psi_{n,r,s} f = F.$$

For $-\pi < \theta < \pi$ this is inverted by

$$F(y,\theta) = \cos^r \theta \, e^{-sy^2 \tan\theta} \, f\left(\frac{y}{\cos\theta}, \tan\theta\right).$$

K acts by $\left(K_{\theta_0} F\right)(y,\theta) = F(y, \theta - \theta_0)$. In the complexified Lie algebra $\mathfrak{sl}(2,\mathbb{C})$,

$$\kappa \equiv i(e^- - e^+) \text{ acts on } I''(n,r,s) \text{ by } i\partial_\theta \tag{3.3}$$

and $\eta^\pm \equiv \frac{1}{2}(h \pm i(e^- + e^+))$ act by

$$\frac{1}{2} e^{\mp 2i\theta}\left[-y\partial_y \mp i\partial_\theta + (r \mp 2isy^2)\right]. \tag{3.4}$$

The Casimir Ω acts by

$$\frac{1}{2} y^2 \widetilde{\nabla}_{n,r,s} + \frac{1}{2} r(r+2) \quad \text{where}$$

$$\widetilde{\nabla}_{n,r,s} \equiv \partial_y^2 - (1+2r)y^{-1}\partial_y + 4s\partial_\theta + 4s^2 y^2.$$

Remark 1. We see an arbitrary $f(x,t) \in C^\infty(\mathbb{R}^2)$ lies in $I'(n,r,s)$ if and only if

$$\lim_{t\to\infty}\left[t^{-r}\, e^{-stx^2}\, f(tx,t)\right] = (-1)^n \lim_{t\to\infty}\left[t^{-r}\, e^{+stx^2}\, f(-tx,-t)\right]$$

exists as a real number and can be used to extend F smoothly (as made precise in Theorem 3) to values of $(\pm 1)^n F(\pm x, \pm\frac{\pi}{2})$.

Lemma 1. *Define the linear injection $M : C^\infty(\mathbb{R}^2) \to C^\infty(\mathbb{R}^2)$ by $(Mu)(\xi,\xi') = \xi^2 u(\xi,\xi')$ for any $u \in C^\infty(\mathbb{R}^2)$. Upon restriction of domain,*

$$M : I'(n,r,s) \to I'(n,r+2,s)$$

is an $SL(2,\mathbb{R})$-intertwining map, $M : I''(n,r,s) \to I''(n,r+2,s)$ is an $SL(2,\mathbb{R})$-intertwining map, and $\psi_{n,r+2,s} \circ M = M \circ \psi_{n,r,s}$ on $I'(n,r,s)$.

Proof. First observe $M : I''(n,r,s) \to I''(n,r+2,s)$ is a well defined map by the definitions. Eq. (3.2) shows $\psi^{-1}_{n,r+2,s} \circ M \circ \psi_{n,r,s} = M$ so that $\psi_{n,r+2,s} \circ M \circ \psi^{-1}_{n,r,s} =$

$M: I'(n,r,s) \to I'(n,r+2,s)$. Finally, Eq. (3.1) implies M intertwines the action of $SL(2,\mathbb{R})$ on $I'(n,r,s)$. $\square$

As a corollary of this, we may view $I'(n,r,s)$ as properly sitting in $I'(n,r+2,s)$ via the inclusion M. This inclusion is compatible with the $SL(2,\mathbb{R})$ action. On the group level, it is straightforward to pull M back from a map $M: I'(n,r,s) \to I'(n,r+2,s)$ to a map (of the same name) $M: I(n,r,s) \to I(n,r+2,s)$ via the restriction map. The resulting map is easily seen to be given by

$$(M\phi)\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, ((v_1,v_2),w)\right) = v_1^2\phi(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, ((v_1,v_2),w))$$

for $\phi \in I(n,r,s)$.

4. The $\ker \nabla_{n,r,s}$, Part I

Definition 1. Let $\ker \nabla_{n,r,s} = \{f \in I'(n,r,s) \mid \nabla_{n,r,s} f = 0\}$ and let $\mathrm{Dom}(\nabla_{n,r,s}) = \{f \in I'(n,r,s) \mid \widetilde{\nabla}_{n,r,s}\psi_{n,r,s} f \in C^\infty(\mathbb{R}^2)\}$.

Theorem 4. (1) *On* $\mathrm{Dom}(\nabla_{n,r,s})$, $\nabla_{n,r,s} = \psi^{-1}_{n,r-2,s}\widetilde{\nabla}_{n,r,s}\psi_{n,r,s}$. *Both* $\ker \nabla_{n,r,s}$ *and* $MI'(n,r-2,s)$ *are subspaces of* $\mathrm{Dom}(\nabla_{n,r,s})$. *If* $F = \psi_{n,r,s} f$, *then* $f \in \ker \nabla_{n,r,s}$ *if and only if* $F \in \ker \widetilde{\nabla}_{n,r,s}$.

(2) $\mathrm{Dom}(\nabla_{n,r,s})$ *is an* $SL(2,\mathbb{R})$*-invariant subspace of* $I'(n,r,s)$ *and*

$$\nabla_{n,r,s} : \mathrm{Dom}(\nabla_{n,r,s}) \to I'(n,r-2,s)$$

is an $SL(2,\mathbb{R})$*-intertwining map. In particular,* $\ker \nabla_{n,r,s}$ *is an* $SL(2,\mathbb{R})$*-invariant subspace of* $I'(n,r,s)$*. Generically,* $\ker \nabla_{n,r,s}$ *is not* $H_3(\mathbb{R})$*-invariant.*

(3) *In the case of* $r = -\frac{1}{2}$, $\mathrm{Dom}(\nabla_{n,-\frac{1}{2},s}) = I'(n,-\frac{1}{2},s)$ *and* $\nabla_{n,-\frac{1}{2},s} : I(n,-\frac{1}{2},s) \to I(n,-\frac{5}{2},s)$ *is in fact a* G*-intertwining map. In particular,* $\ker \nabla_{n,-\frac{1}{2},s}$ *is* G*-invariant.*

Proof. Begin with relation $\psi_{n,r,s} M\nabla_{n,r,s} = \psi_{n,r,s}(2\Omega - r(r+2)) = (2\Omega - r(r+2))\psi_{n,r,s} = M\widetilde{\nabla}_{n,r,s}\psi_{n,r,s}$. In particular, if $f_\nabla \in \ker \nabla_{n,r,s}$, then $M\widetilde{\nabla}_{n,r,s}\psi_{n,r,s} f_\nabla = 0$ so that $\widetilde{\nabla}_{n,r,s}\psi_{n,r,s} f_\nabla = 0$ and $\ker \nabla_{n,r,s} \subseteq \mathrm{Dom}(\nabla_{n,r,s})$. If $f_{r-2} \in I'(n,r-2,s)$, then $\widetilde{\nabla}_{n,r,s}\psi_{n,r,s} M f_{r-2} = \widetilde{\nabla}_{n,r,s} M \psi_{n,r-2,s} f_{r-2} \in C^\infty(\mathbb{R}^2)$. Thus $MI'(n,r-2,s) \subseteq \mathrm{Dom}(\nabla_{n,r,s})$.

More generally observe that if $f \in \mathrm{Dom}(\nabla_{n,r,s})$, then we may in fact view $\widetilde{\nabla}_{n,r,s}\psi_{n,r,s} f \in I''(n,r-2,s)$. This allows the use of Lemma 1 on the relation $M\nabla_{n,r,s} f = \psi^{-1}_{n,r,s} M\widetilde{\nabla}_{n,r,s}\psi_{n,r,s} f$ to get $M\nabla_{n,r,s} f = M\psi^{-1}_{n,r-2,s}\widetilde{\nabla}_{n,r,s}\psi_{n,r,s} f$. Thus $\nabla_{n,r,s} = \psi^{-1}_{n,r-2,s} \widetilde{\nabla}_{n,r,s}\psi_{n,r,s}$ on $\mathrm{Dom}(\nabla_{n,r,s})$ and so $\nabla_{n,r,s} : \mathrm{Dom}(\nabla_{n,r,s}) \to I'(n,r-2,s)$. To check invariance and intertwining properties, let $g \in SL(2,\mathbb{R})$ and write g_r to indicate the action of g on $I''(n,r,s)$. Using similar techniques as above, $M\widetilde{\nabla}_{n,r,s}\psi_{n,r,s}g_r f = M\widetilde{\nabla}_{n,r,s}g_r\psi_{n,r,s}f = g_r M\widetilde{\nabla}_{n,r,s}\psi_{n,r,s}f = Mg_{r-2}\widetilde{\nabla}_{n,r,s}\psi_{n,r,s}f$ so that $\widetilde{\nabla}_{n,r,s}\psi_{n,r,s}g_r f = g_{r-2}\widetilde{\nabla}_{n,r,s}\psi_{n,r,s}f \in C^\infty(\mathbb{R}^2)$ and $\mathrm{Dom}(\nabla_{n,r,s})$ is $SL(2,\mathbb{R})$-invariant. Applying $\psi^{-1}_{n,r-2,s}$ to both sides shows $\nabla_{n,r,s}$ is an $SL(2,\mathbb{R})$-intertwining map.

Regarding the statements on $H_3(\mathbb{R})$, an easy calculation shows

$$[\nabla_{n,r,s},\,(-v_1+v_2t)\partial_x+(-2sv_2x+\omega s)]=(1+2r)\left[2sv_2x^{-1}+(-v_1+v_2t)x^{-2}\partial_x\right]$$

so $\ker\nabla_{n,r,s}$ is $H_3(\mathbb{R})$-invariant when $r=-\frac{1}{2}$. To examine the converse, let $h_y = (I,((y,0),0)) \in H_3(\mathbb{R})$. It is easy to check that if $f\in\ker\nabla_{n,r,s}$, then $\left[\nabla_{n,r,s}(h_y f)\right](x,t) = (1+2r)\frac{y}{x(x-y)}\left(\partial_x f\right)(x-y,t)$. In particular if $r\neq -\frac{1}{2}$, $h_y f\in\ker\nabla_{n,r,s}$ if and only if $\partial_x f=0$. However, we will see below (Theorems 6 and 7) there generically exist solutions to $\nabla_{n,r,s}f=0$ in $I'(n,r,s)$ with nontrivial dependence on x. $\square$

Definition 2. Write

$$\ker\nabla^{\pm}_{n,r,s}=\{f\in\ker\nabla_{n,r,s}\mid f(-x,t)=\pm f(x,t)\}.$$

It is easy to see $\ker\nabla^{\pm}_{n,r,s}$ is also $SL(2,\mathbb{R})$-invariant and $\ker\nabla_{n,r,s}=\ker\nabla^{+}_{n,r,s}\oplus\ker\nabla^{-}_{n,r,s}$.

Since $\nabla_{n,r,s}\equiv\partial_x^2-(1+2r)x^{-1}\partial_x+4s\partial_t$, it is possible to explicitly consider a change of variables in the x-coordinate.

Theorem 5. *For $s\in\mathbb{C}\backslash\{0\}$ there is an $SL(2,\mathbb{R})$-intertwining isomorphism $T':I'(n,r,s)\to I'(n,r,\frac{s}{|s|})$ that restricts to an isomorphism from $\ker\nabla^{\pm}_{n,r,s}$ to $\ker\nabla^{\pm}_{n,r,\frac{s}{|s|}}$. The map T' is given by $(T'f)(x,t)=f(\frac{x}{\sqrt{|s|}},t)$.*

Proof. For $g=(A,(v,w))\in G$ and $c\in\mathbb{R}\backslash\{0\}$, define $c\cdot g=(A,(cv,c^2w))$. From the group structure on G, it is easy to see $c\cdot(gg')=(c\cdot g)(c\cdot g')$ for all $g,g'\in G$. If we define $T_c\phi$ by $(T_c\phi)(g)=\phi(c\cdot g)$ for $\phi\in I(n,r,s)$, it follows from the definition of $I(n,r,s)$ that $T_c\phi\in I(n,r,c^2s)$. The map T_c is clearly an $SL(2,\mathbb{R})$-intertwining isomorphism with inverse $T_{c^{-1}}$, though it does not respect the action of $H_3(\mathbb{R})$. In terms of $f\in I'(n,r,s)$, it follows by restriction that the corresponding map T'_c acts as $(T'_c f)(x,t)=f(cx,t)$. A simple change of variables immediately implies $T_c\nabla_{n,r,s}T_c^{-1}=c^{-2}\nabla_{n,r,c^2s}$ so that T'_c maps $\ker\nabla^{\pm}_{n,r,s}$ onto $\ker\nabla^{\pm}_{n,r,c^2s}$. Putting $c=|s|^{-\frac{1}{2}}$ finishes the proof. $\square$

More generally, a similar argument and use of analytic extension allows one to show $I'(n,r,s)$ is independent of s when $s\neq 0$. However, we will see this directly in Section 5.

Recall one of the standard ways to realize the representations of $SL(2,\mathbb{R})$ is on $\mathcal{S}^{n,r}$, the set of even or odd (depending on the parity of $n\in\mathbb{Z}$) smooth functions on $\mathbb{R}^2\backslash\{0\}$ homogenous of degree $r \in \mathbb{C}$ under positive dilation (e.g., [4, §III.1.2]). Irreducibility, equivalence, and unitarity issues are all well known. In particular, note $\mathcal{S}^{n,r}$ is infinitesimally equivalent to $\mathcal{S}^{n,-2-r}$ except when $\mathcal{S}^{n,r}$ is reducible, *i.e.*, except when $r \in \mathbb{Z}$ and $n \equiv r \mod(2)$. It is also known that $\mathcal{S}^{n,r}$ has a basis $\{f_m\}$ where $m \equiv n \operatorname{mod}(2)$ satisfying

$$\kappa f_m = m f_m,$$

$$\eta^{\pm} f_m = \frac{1}{2}(-r \pm m) f_{m\pm 2}. \tag{4.1}$$

The case of $s = 0$ is simple and is examined in the theorem below.

Theorem 6. (1) *For* $r \notin \{-\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\}$, $\ker \nabla_{n,r,0} = \ker \nabla^{+}_{n,r,0}$ *and is infinitesimally isomorphic to* $\mathcal{S}^{n,r}$ *as an* $SL(2,\mathbb{R})$ *representation.* $H_3(\mathbb{R})$ *acts trivially on* $\ker \nabla_{n,r,0}$.

(2) *For* $r \in \{-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \ldots\}$, $\ker \nabla_{n,r,0} = \ker \nabla^{+}_{n,r,0} \oplus \ker \nabla^{-}_{n,r,0}$ *with* $\ker \nabla^{+}_{n,r,0}$ *infinitesimally isomorphic to* $\mathcal{S}^{n,r}$ *and* $\ker \nabla^{-}_{n,r,0}$ *infinitesimally isomorphic to* $\mathcal{S}^{n+1,-r-2}$ *as* $SL(2,\mathbb{R})$ *representations. Though* $H_3(\mathbb{R})$ *acts trivially on the* $\mathcal{S}^{n,r}$ *component of* $\ker \nabla_{n,r,0}$, *it can map the* $\mathcal{S}^{n+1,-r-2}$ *component outside the* $\ker \nabla_{n,r,0}$.

(3) *For* $r \in \{0, 1, 2, \ldots\}$, $\ker \nabla_{n,r,0} = \ker \nabla^{+}_{n,r,0}$ *and is infinitesimally isomorphic to* $\mathcal{S}^{n,r} \oplus \mathcal{S}^{n,-r-2}$ *as an* $SL(2,\mathbb{R})$ *representation. Though* $H_3(\mathbb{R})$ *acts trivially on the* $\mathcal{S}^{n,r}$ *component of* $\ker \nabla_{n,r,0}$, *it can map the* $\mathcal{S}^{n+2r,-r-2}$ *component outside the* $\ker \nabla_{n,r,0}$.

Proof. When $s = 0$, $\widetilde{\nabla}_{n,r,0} \equiv (\partial_y - (1+2r)y^{-1})\partial_y$. The general solution to $(\partial_y - (1+2r)y^{-1})\partial_y F(y,\theta) = 0$ is $F(y,\theta) = \psi_1(\theta) + y^{2+2r}\psi_2(\theta)$ when $r \neq -1$ and $F(y,\theta) = \psi_1(\theta) + \ln y\, \psi_2(\theta)$ when $r = -1$. For $r \notin \{-\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\}$, smoothness of F shows $\ker \widetilde{\nabla}_{n,r,0} = \{F(y,\theta) = \psi(\theta) \mid \psi \in C^{\infty}(\mathbb{R}) \text{ with } \psi(\theta+\pi) = (-1)^n \psi(\theta)\}$. As $\left(K_{\theta_0} F\right)(y,\theta) = F(y, \theta - \theta_0)$, the K-finite vectors are $F_m(y,\theta) = e^{-im\theta}$ where $m \in \mathbb{Z}$ with $m \equiv n \operatorname{mod}(2)$. It is easy to check that Theorem 3 gives

$$\kappa F_m = m F_m,$$

$$\eta^{\pm} F_m = \frac{1}{2}\,(r \mp m)\, F_{m\pm 2}.$$

In particular, $\ker \widetilde{\nabla}_{n,r,0}$ is infinitesimally equivalent to $\mathcal{S}^{n,r}$.

Pulling everything back with Eq. (3.2), we see $\ker \nabla_{n,r,0} = \{f(x,t) = (1+t^2)^{\frac{r}{2}}\psi(\arctan t) \mid \psi \in C^{\infty}(\mathbb{R}),\ \psi(\theta+\pi) = (-1)^n \psi(\theta)\} = \ker \nabla^{+}_{n,r,0}$. Since

$$(hf)(x,t) = f(x - v_1 + t v_2, t) \tag{4.2}$$

for $h = ((v_1, v_2), w) \in H_3(\mathbb{R})$, the action of $H_3(\mathbb{R})$ does not affect the t variable. It follows that $H_3(\mathbb{R})$ acts trivially on $\ker \nabla_{n,r,0}$.

For $r \in \{-\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\}$, $\ker \widetilde{\nabla}_{n,r,0} = \{F(y,\theta) = \psi_1(\theta) + y^{2+2r}\psi_2(\theta) \mid \psi_i \in C^\infty(\mathbb{R}),\ \psi_1(\theta+\pi) = (-1)^n\psi_1(\theta),\ \psi_2(\theta+\pi) = (-1)^{2r+n}\psi_2(\theta)\}$. The K-finite vectors are F_m, $m \equiv n \bmod (2)$, as above and $\widetilde{F}_{\widetilde{m}}(y,\theta) = y^{2+2r}e^{-i\widetilde{m}\theta}$, $\widetilde{m} \equiv n + 2r \bmod (2)$. It is again straightforward to very that Theorem 3 gives

$$\kappa \widetilde{F}_{\widetilde{m}} = \widetilde{m}\,\widetilde{F}_{\widetilde{m}},$$

$$\eta^{\pm}\widetilde{F}_{\widetilde{m}} = \frac{1}{2}\,(-r-2\mp\widetilde{m})\,\widetilde{F}_{\widetilde{m}\pm 2}.$$

In particular, $\ker \widetilde{\nabla}_{n,r,0}$ is infinitesimally equivalent to $\mathcal{S}^{n,r} \oplus \mathcal{S}^{n+2r,-r-2}$.

Pulling everything back with Eq. (3.2), we see $\ker \nabla_{n,r,0} = \{f(x,t) = (1+t^2)^{\frac{r}{2}}\psi_1(\arctan t) + x^{2+2r}(1+t^2)^{-\frac{r+2}{2}}\psi_2(\arctan t) \mid \psi_1 \in C^\infty(\mathbb{R}),\ \psi_1(\theta+\pi) = (-1)^n\psi_1(\theta),\ \psi_2(\theta+\pi) = (-1)^{2r+n}\psi_2(\theta)\}$. Then $\mathcal{S}^{n,r}$ is associated to $\{f(x,t) = (1+t^2)^{\frac{r}{2}}\psi_1(\arctan t) \mid \psi_1 \in C^\infty(\mathbb{R}),\ \psi_1(\theta+\pi) = (-1)^n\psi_1(\theta)\}$ and is contained in $\ker \nabla^+_{n,r,0}$. Similarly, $\mathcal{S}^{n+2r,-r-2}$ is associated to $\{f(x,t) = x^{2+2r}(1+t^2)^{-\frac{r+2}{2}}\psi_2(\arctan t) \mid \psi_2 \in C^\infty(\mathbb{R}),\ \psi_2(\theta+\pi) = (-1)^{2r+n}\psi_2(\theta)\}$ and is contained in $\ker \nabla^+_{n,r,0}$ if $r \in \{0,1,2,\ldots\}$ and in $\ker \nabla^-_{n,r,0}$ if $r \in \{-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \ldots\}$. Eq. (4.2) again shows that $H_3(\mathbb{R})$ acts trivially on the $\mathcal{S}^{n,r}$ component of $\ker \nabla_{n,r,0}$. Similarly, Eq. (4.2) shows that $H_3(\mathbb{R})$ can map the $\mathcal{S}^{n+2r,-r-2}$ component out of $\ker \nabla_{n,r,0}$ outside of $\ker \nabla_{n,r,0}$; e.g., if $f(x,t) = x^{2+2r}(1+t^2)^{-\frac{r+2}{2}}$, then $((0,1),0)f)(x,t) = (x+t)^{2+2r}(1+t^2)^{-\frac{r+2}{2}} \notin \ker \nabla_{n,r,0} = \ker[\partial_x^2 - (1+2r)x^{-1}\partial_x]$. $\square$

5. The $\ker \nabla_{n,r,s}$, Part II

In this section, we assume $s \neq 0$. For $f \in I'(n,r,s)$, we continue to employ the notation $F = \psi_{n,r,s} f \in I''(n,r,s)$ for the corresponding function as in Eq. (3.2).

We first look for K-finite vectors in $\ker \widetilde{\nabla}_{n,r,s}$. By the representation theory of the circle group S^1, it follows that the K-finite functions are spanned by functions F_m satisfying $K_\theta F_m = e^{im\theta}F_m$ for some $m \in \mathbb{Z}$ and $F_m \in I''(n,r,s)$. Since Theorem 3 says $\left(K_{\theta_0}F_m\right)(y,\theta) = F_m(y,\theta-\theta_0)$, $F_m(y,\theta)$ must be of the form $e^{-im\theta}h_m(y)$ for some smooth h_m. The parity condition $F_m(y,\theta+\pi) = (-1)^n F_m(-y,\theta)$ reduces to $(-1)^m h_m(y) = (-1)^n h_m(-y)$. Substituting $e^{-im\theta}h_m$ in for F_m, an elementary calculation shows that $F_m = e^{-im\theta}h_m \in \ker \widetilde{\nabla}_{n,r,s}$ if and only if $h''_m - (1+2r)y^{-1}h'_m + (-4ism + 4s^2y^2)h_m = 0$.

To summarize, we see that the K-finite vectors in $\ker \widetilde{\nabla}_{n,r,s}$ are spanned by functions of the form $F_m(y,\theta) = e^{-im\theta}h_m(y)$ where h_m is smooth, satisfies the *parity condition*

$$h_m(-y) = (-1)^{m+n}h_m(y)$$

and solves the differential equation

$$h''_m - (1+2r)y^{-1}h'_m + (-4ism + 4s^2y^2)h_m = 0. \tag{5.1}$$

We say that h_m *exists* if there exists a smooth function, not identically zero, satisfying the above parity condition and differential equation. We will see below (Lemma 2) that *generically* h_m exists if and only if $m \equiv n \bmod (2)$ and that h_m is unique up to a scalar multiplication in this case. However in singular cases, there may be two independent solutions. When this happens we will identify two independent solutions below and name them h_m and $\widetilde{h}_m$.

When such an h_m exists, write $F_m = e^{-im\theta} h_m$ for the corresponding K-finite vector in $\ker \widetilde{\nabla}_{n,r,s}$ and write f_m for the corresponding function to F_m in $I'(n, r, s)$. Similar notation is used for $\widetilde{h}_m$, $\widetilde{F}_m$, and $\widetilde{f}_m$ when they exist. From Eq. (3.2), it is known $f_m(x, t) = (1+t^2)^{\frac{r}{2}} e^{\frac{stx^2}{1+t^2}} F_m(\frac{x}{\sqrt{1+t^2}}, \arctan t)$. Since $(1+t^2)^{\frac{r}{2}} e^{-im \arctan t} = (1+t^2)^{\frac{r+m}{2}} (1+it)^{-m}$, which we write for the sake of symmetry imprecisely as $(1-it)^{\frac{r+m}{2}} (1+it)^{\frac{r-m}{2}}$, we obtain

$$f_m(x, t) = (1+it)^{\frac{r-m}{2}} (1-it)^{\frac{r+m}{2}} e^{\frac{stx^2}{1+t^2}} h_m\left(\frac{x}{\sqrt{1+t^2}}\right) \tag{5.2}$$

with similar results for $\widetilde{f}_m$ when it exists.

Note by the Stone–Weierstrass Theorem that the $\mathrm{sp}_{\mathbb{C}}\{e^{-im\theta} h(y) \mid m \in \mathbb{Z}, h \in C^\infty(\mathbb{R})\}$ is dense in $C^\infty(\mathbb{R} \times S^1)$ with respect to the C^∞-topology. In particular, any $F \in I''(n, r, s)$ may be written as $F(y, \theta) = \sum_{m \in \mathbb{Z}} e^{-im\theta} h_m(y)$ for some $h_m \in C^\infty(\mathbb{R})$ where the sum (and its derivatives) converges uniformly on compact sets. It follows that $h_m(-y) = (-1)^{m+n} h_m(y)$ and that $F \in \ker \nabla_{n,r,s}$ if and only if each h_m satisfies Eq. (5.1). In particular, the K-finite vectors are dense in $\ker \widetilde{\nabla}_{n,r,s}$.

Recall also the principal series representations (normalized, smooth, noncompact picture) of $SL(2, \mathbb{R})$ may be also realized as $\mathcal{P}^{n,r} = L^2(\mathbb{R}, (1+t^2)^{\mathrm{Re}\, r}\, dt)$ for $n \in \mathbb{Z}$, depending only on the parity of n, and $r \in \mathbb{C}$ (e.g., [5, §II.5]). The action is given by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \psi\right)(t) = \mathrm{sgn}^n(-ct+a)\, |-ct+a|^{-r-1}\, \psi\left(\frac{dt-b}{-ct+a}\right) \tag{5.3}$$

for $\psi \in \mathcal{P}^{n,r}$ and we write

$$\|\psi\|_{\mathcal{P}^{n,r}} = \left[\int_{-\infty}^{\infty} |\psi(t)|^2 (1+t^2)^{\mathrm{Re}\, r} dt\right]^{\frac{1}{2}}.$$

To compare with previous notation, note by restricting $\mathcal{S}^{n,r}$ to the unit circle, that $\mathcal{S}^{n,r}$ becomes the infinitessimal compact picture of $\mathcal{P}^{n,-r-1}$. Moreover, it is well known that the irreducible finite $(r+1)$-dimensional $SL(2, \mathbb{R})$ representation $V_r = \{$polynomials of degree at most $r\} \subseteq \mathcal{P}^{r,-r-1}$ (e.g., [5] p. 38) for $r \in \mathbb{N}$.

Lemma 2. *Fix* $s \neq 0$.

(1) *For* $r \notin \{-\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots\}$, h_m *exists if and only if* $m \equiv n \bmod (2)$. *It can be normalized so* $h_m(0) = 1$ *after which it is unique.*
(2) *For* $r \in \{-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots\}$, h_m *exists for each* $m \in \mathbb{Z}$.
   (a) *If* $m \equiv n$, h_m *can be normalized so* $h_m(0) = 1$ *after which it is unique.*
   (b) *If* $m \equiv n + 1$, *it can be normalized so* $h_m^{(2r+2)}(0) = 1$ *after which it is unique. In this case,* $h_m^{(k)}(0) = 0$ *for all* $0 \leqslant k < 2r + 2$.
(3) *For* $r \in \{0, 1, 2, \ldots\}$, h_m *exists if and only if* $m \equiv n$.
   (a) *If* $r \equiv m + 1$, *then* h_m *can be normalized so* $h_m^{(2r+2)}(0) = 1$ *after which it is unique.*
   (b) *If* $r \equiv m$, *then the solution space for* h_m *is one dimensional when* $|m| > r$ *and is two dimensional when* $|m| \leqslant r$. *For any* $m \in \mathbb{Z}$, *one solution, denoted* h_m, *is uniquely determined by the conditions* $h_m(0) = 0$ *and* $h_m^{(2r+2)}(0) = 1$. *In this case,* $h_m^{(k)}(0) = 0$ *for all* $0 \leqslant k < 2r + 2$. *When* $|m| \leqslant r$, *there is a second independent solution, denoted* $\widetilde{h}_m$, *that can be normalized so* $\widetilde{h}_m(0) = 1$ *and which is then determined only up to adding a multiple of* h_m.

Proof. Using standard techniques for regular singular ordinary differential equations, most of the statements are routine to check (e.g., [1]). The main limiting factors for the existence of h_m will be smoothness and parity. The only real difficulty will be with the existence of $\widetilde{h}_m$ in part (3b). As usual in the theory of regular singular ordinary differential equations, substituting $h_m = \sum_{k=0}^{\infty} c_{k,m}(\lambda) y^{k+\lambda}$ into Eq. (5.1) for h_m quickly gives the recurrence relation

$$(\lambda + k)(\lambda + k - 2 - 2r)\, c_{k,m}(\lambda) - 4ism\, c_{k-2,m}(\lambda) + 4s^2\, c_{k-4,m}(\lambda) = 0. \tag{5.4}$$

Hence the roots of the indicial equation, i.e., the exponents of singularity, are 0 and $2r + 2$. When $2r + 2 \notin \mathbb{Z}$, it is known there are two independent solutions to Eq. (5.1) of the form $h_m(y) = 1 + \sum_{k=1}^{\infty} c_{2k} y^{2k}$ and $y^{2r+2}(1 + \sum_{k=1}^{\infty} d_{2k} y^{2k})$, respectively, for some constants c_{2k} and d_{2k}. If $2r + 2 \in \mathbb{Z}_{<0}$, then it is known there are two independent solutions of the form $h_m(y) = 1 + \sum_{k=1}^{\infty} c_{2k} y^{2k}$ and $a(m) h_m(y) \ln|y| + y^{2r+2}(1 + \sum_{k=1}^{\infty} c_{2k} y^{2k})$, respectively, for some constant $a(m)$. If $2r + 2 = 0$, then it is known that independent solutions to Eq. (5.1) have the form $h_m(y) = 1 + \sum_{k=1}^{\infty} c_{2k} y^{2k}$ and $h_m(y) \ln|y| + (1 + \sum_{k=1}^{\infty} d_{2k} y^{2k})$, respectively. In any of these cases, only the first solution is smooth and it has the correct parity if and only if $m + n \equiv 0$. Part (1) follows.

If $2r+2 \in \mathbb{Z}_{>0}$, the two independent solutions to Eq. (5.1) have the form $\frac{1}{(2r+2)!} y^{2r+2}(1 + \sum_{k=1}^{\infty} c_{2k} y^{2k})$ and $a(m) y^{2r+2}(1 + \sum_{k=1}^{\infty} c_{2k} y^{2k}) \ln|y| + (1 + \sum_{k=1}^{\infty} d_{2k} y^{2k})$ for some constant $a(m)$. In particular, the second solution is smooth if and only if $a(m) = 0$. It is known that $a(m) = \lim_{\lambda \to 0} \left(\lambda c_{2r+2,m}(\lambda)\right)$ where $c_{k,m}(\lambda)$ is determined by the recurrence relation and the initial conditions $c_{0,m}(\lambda) = 1$ and $c_{k,m}(\lambda) = 0$ for $k \in \mathbb{Z}_{<0}$. Since the recurrence relation in Eq. (5.4) implies $c_{k,m}(\lambda) = 0$ when k is odd, it follows that $a(m) = 0$ trivially when $2r + 2$ is odd. Thus when $2r$ is odd, there are two smooth independent solutions. However, the parity condition is satisfied only by the first solution when $m+n \equiv 1$ and only by the second solution when $m+n \equiv 0$. In each case, the solution is called h_m and part (2) follows.

If $2r+2 \in \mathbb{Z}_{>0}$ and $2r+2$ is even, then the parity condition gives no solutions when $m+n \equiv 1$ and one or two independent solutions when $m+n \equiv 0$. Depending on the value of $a(m)$, the smoothness condition may eliminate the second solution. We write h_m for the first solution and $\widetilde{h}_m$ for the second when it exists, i.e., when $a(m) = 0$. Unfortunately, calculating $a(m)$ directly can be quite difficult for large r. Nevertheless, using the recursive definition of $c_{k,m}(\lambda)$, it follows that $a(m) = \lim_{\lambda\to 0}\left(\lambda c_{2r+2,m}(\lambda)\right)$ is a polynomial of degree $r+1$ in m. We will show below that $a(m) = 0$ when $m \in \{-r, -r+2, \ldots, r-2, r\}$. By the degree of $a(m)$, it will follow that $a(m) = 0$ if and only if $m \in \{-r, -r+2, \ldots, r-2, r\}$ and part (3) will follow.

To show $a(m) = 0$ when $m \in \{-r, -r+2, \ldots, r-2, r\}$, we first show that $a(0) = 0$ when r is even and that $a(1) = 0$ when r is odd. What is easy to see from the recurrence relation is that when $m = 0$ then $c_{k,0}(\lambda) = 0$ for $k \notin 4\mathbb{Z}_{\geqslant 0}$ so that $c_{2r+2,0}(\lambda) = 0$ when r is even. Thus $a(0) = \lim_{\lambda\to 0}\left(\lambda c_{2r+2,0}(\lambda)\right) = 0$ when r is even.

For the case of $m = 1$ and r odd, a bit more work is needed. First define

$$d_k(\lambda) = \frac{(\lambda+k)(\lambda+k-2)\cdots(\lambda+2)\,(\lambda+k-2-2r)(\lambda+k-4-2r)\cdots(\lambda-2r)}{(4is)^{\frac{k}{2}}} \times c_{k,1}(\lambda).$$

so that the recurrence relation from Eq. (5.4) reduces to

$$d_k(\lambda) = d_{k-2}(\lambda) + 4^{-1}(\lambda+k-2)(\lambda+k-4-2r)d_{k-4}. \tag{5.5}$$

In particular, $d_k(\lambda)$ is polynomial in λ so that

$$\lim_{\lambda\to 0}\left(\lambda c_{2r+2,0}(\lambda)\right) = \frac{(4is)^{r+1}}{2^{r+1}(r+1)!(-2)^r r!}\lim_{\lambda\to 0} d_{2r+2}(\lambda) = \frac{(-1)^r(4is)^{r+1}}{2^{2r+1}(r+1)!r!}d_{2r+2}(0).$$

Thus $a(1) = 0$ if and only if $d_{2r+2}(0) = 0$. Straightforward induction on Eq. (5.5) shows

$$d_{4j}(0) = \left[(1-r)(3-r)\cdots(\mathbf{2j-1-r})\right]\left[(2j-1)(2j-3)\cdots(1)\right],$$

$$d_{4j+2}(0) = \left[(1-r)(3-r)\cdots(2j-1-r)\right]\left[(2j+1)(2j-1)\cdots(1)\right].$$

If r is odd, then $r = 2j-1$ for some j. In this case, $d_{2r+2}(0) = d_{4j}(0) = 0$ as desired.

Hence $\widetilde{h}_0$ and $\widetilde{h}_1$ exist when r is even and odd, respectively. To see how $\widetilde{h}_0$ or $\widetilde{h}_1$ force the existence of the remaining $\widetilde{h}_m$, we make use of the $SL(2,\mathbb{R})$ action. It is straightforward to check (in fact, see the general statement in Lemma 4 below) with Eqs. (3.1) and (5.3) that the map $T_0 : I'(r,r,s) \to \mathcal{P}^{r,-r-1}$ given by $T_0(f) = f(0,\cdot)$

intertwines the $SL(2,\mathbb{R})$ action. With an eye towards Eq. (5.2), it is also easy to check that if $r \in \{0,1,2,\ldots\}$ then $(1+it)^{\frac{r-m}{2}}(1-it)^{\frac{r+m}{2}} = (1+t^2)^{\frac{r+m}{2}}(1+it)^{-m} = (1+t^2)^{\frac{r-m}{2}}(1-it)^m$ is a polynomial (of degree r) if and only if $m \in \{-r,-r+2,\ldots,r-2,r\}$.

The existence of $\widetilde{h}_m$ for $m \in \{-r,-r+2,\ldots,r-2,r\}$ will now be established using induction on m starting with m equal to 0 or 1, depending on the parity of r. To see $\widetilde{h}_{m\pm2}$ exists for $|m\pm 2| \leqslant r$, observe, by Eq. (5.2) and the observations made in the previous paragraph, that $T_0\widetilde{f}_m$ is a nonzero element of V_r since $\widetilde{h}_m(0)=1$. On the other hand, elementary $SL(2,\mathbb{R})$ theory implies $\eta^{\pm}\widetilde{f}_m$ has κ-weight $m\pm2$. Thus $\eta^{\pm}\widetilde{f}_m$ must either be a multiple of $f_{m\pm2}$ or a linear combination of $f_{m\pm2}$ and $\widetilde{f}_{m\pm2}$ if $\widetilde{h}_{m\pm2}$ exists. But since $\eta^{\pm}T_0\widetilde{f}_m$ is nonzero by the structure of V_r, it follows that $T_0\eta^{\pm}\widetilde{f}_m$ is nonzero. As $T_0 f_{m\pm2}=0$, $\widetilde{h}_{m\pm2}$ must exist. $\square$

Special consideration is due case (3b) of Lemma 2. We remark in this case, e.g. $r \in \mathbb{N}$ and $m \equiv n \equiv r \mod (2)$, it is possible to explicitly choose nice representatives of $\widetilde{h}_m$ and $\widetilde{F}_m$ for $|m| \leqslant r$ as well as for h_m and F_m for any $m \equiv n$.

In particular, for the special values of $m=\pm r$, $\eta^{\pm}$ maps $\mathrm{sp}_{\mathbb{C}}\{F_{\pm r}, \widetilde{F}_{\pm r}\}$ to a multiple of $F_{\pm(r+2)}$ so that $\eta^{\pm}$ has a nontrivial kernel in $\mathrm{sp}_{\mathbb{C}}\{F_{\pm r}, \widetilde{F}_{\pm r}\}$. We write elements of this kernel as $e^{\mp ir\theta}h_{\pm}(y)$. Eq. (3.4) gives $\eta^{\pm} = \frac{1}{2}e^{\mp 2i\theta}\left[-y\partial_y \mp i\partial_\theta + (r \mp 2isy^2)\right]$ so that $\eta^{\pm}\left[e^{\mp ir\theta}h_{\pm}(y)\right]=0$ if and only if $h'_{\pm} = \mp 2isyh_{\pm}$. Up to a scalar multiple, the only solution to this first-order differential equation is $h_{\pm} = e^{\mp isy^2}$. Now induction easily implies

$$\begin{aligned}&\left(\eta^{\mp}\right)^k\left[e^{\mp ir\theta}e^{\mp isy^2}\right]\\ &\quad = \sum_{j=0}^{k}\binom{k}{j}\left(\prod_{l=j}^{k-1}(r-l)\right)\left(\pm 2isy^2\right)^j\ e^{\mp i(r-2k)\theta}e^{\mp isy^2}\end{aligned} \tag{5.6}$$

for $k \in \mathbb{N}$. In particular, the polynomial factor of $e^{\mp i(r-2k)\theta}e^{\mp isy^2}$ in Eq. (5.6) is even and of degree $2k$. When $0 \leqslant k \leqslant r$, the polynomial has nonzero coefficients exactly in degrees $0,2,\ldots,2k$. However, when $k>r$, the polynomial has nonzero coefficients exactly in degrees $2(r+1),2(r+2),\ldots,2k$. These polynomials are related to the classically defined *Laguerre* polynomials (e.g., see [6]).

Define $G^{\mp}_{\pm(r-2k)} = \left(\eta^{\mp}\right)^k\left[e^{\mp ir\theta}e^{\mp isy^2}\right]$ for $k \in \mathbb{N}$ so that

$$G^{\pm}_m(y,\theta) = \sum_{j=0}^{\frac{r\pm m}{2}}\binom{\frac{r\pm m}{2}}{j}\left(\prod_{l=j}^{\frac{r\pm m}{2}-1}(r-l)\right)\left(\mp 2isy^2\right)^j\ e^{im\theta}e^{\pm isy^2} \quad \text{for } \pm m \geqslant -r. \tag{5.7}$$

In this terminology, the polynomial factor of $e^{im\theta}e^{\pm isy^2}$ in $G^{\pm}_m$ is even of degree $r \pm m$. When $|m| \leqslant r$ the polynomial has nonzero coefficients exactly in degrees

$0, 2, \ldots (r \pm m)$. When $|m| > r$, the polynomial has nonzero coefficients exactly in degrees $2(r+1), 2(r+2), \ldots (r \pm m)$.

When $|m| > r$, Lemma 2 therefore implies $G_m^{\pm}$ is a nonzero multiple of F_m and so $\{F_m \mid |m| > r\}$ is explicitly identified up to an easily calculable constant.

For $|m| \leqslant r$, evaluating Eq. (5.7) at $y = \theta = 0$ gives $G_m^{\pm}(0,0) = \frac{r!}{(\frac{r \pm m}{2})!}$ so that by Lemma 2 we have $G_m^{\pm} = \frac{r!}{(\frac{r \pm m}{2})!} \widetilde{F}_m + c_m^{\pm} F_m$ for some constant $c_m^{\pm}$. For $|m| \leqslant r$, $\{\widetilde{F}_m \mid |m| \leqslant r\}$ is therefore explicitly known up to addition of a multiple of F_m. Furthermore, $\{G_m^+, G_m^-\}$ is clearly independent due to the presence of the $e^{\mp isy^2}$ term (recall $s \neq 0$ here) in Eq. (5.7). Thus Lemma 2 says $\mathrm{sp}_{\mathbb{C}}\{G_m^+, G_m^-\} = \mathrm{sp}_{\mathbb{C}}\{F_m, \widetilde{F}_m\}$. Since $\left(\frac{r+m}{2}\right)! G_m^+(0,0) - \left(\frac{r-m}{2}\right)! G_m^-(0,0) = 0$, Lemma 2 forces $\left(\frac{r+m}{2}\right)! G_m^+ - \left(\frac{r-m}{2}\right)! G_m^-$ to be a multiple of F_m. Independence shows it is a nonzero multiple. Thus $\{F_m \mid |m| \leqslant r\}$ is also explicitly identified up to a calculable normalization.

Definition 3. In the case of $r \in \mathbb{N}$ and $m \equiv n \equiv r \bmod (2)$, define $V_r^{\pm}$ to be the closure of $\mathrm{sp}_{\mathbb{C}}\{G_m^{\pm} \mid \pm m \geqslant -r\}$ with respect to the C^{∞}-topology.

The above discussion and $SL(2,\mathbb{R})$ theory establishes the following lemma.

Lemma 3. *Let $s \neq 0$. In the case of $r \in \mathbb{N}$ and $n \equiv r \bmod (2)$, $\ker \widetilde{\nabla}_{n,r,s} = V_r^+ \oplus V_r^-$. Moreover, V_r^+ is an indecomposible lowest (highest, respectively, for V_r^-) $SL(2,\mathbb{R})$ module. The only $SL(2,\mathbb{R})$ invariant subspace of $V_r^{\pm}$ is $W_r^{\pm} = \overline{\mathrm{sp}_{\mathbb{C}}\{G_m^{\pm} \mid \pm m \geqslant r,\ m \equiv n\}}$, the quotient of which is isomorphic to V_r.*

Observe there is a linear bijection (*non*-intertwining) on $\ker \widetilde{\nabla}_{n,r,s}$ that permutes V_r^+ and V_r^- and is induced by the change of variables $F(y,\theta) \to F(iy, -\theta)$. Proper sense of this can be made by analytic continuation. Second, it should be noted that $W_r^{\pm}$ is infinitesimally isomorphic to a discrete series representation (Theorem 7) and exhausts all such. It is well known (e.g., [5, p. 38]) that the discrete series sit as subrepresentations in $\mathcal{P}^{r,r-1} \cong \mathcal{S}^{r,-r}$, $r \in \{2,3,\ldots\}$. Looking at $m = \pm(r+2)$ in Eq. (5.7), the extremal vectors of $W_r^{\pm}$ are especially simple and, up to scalar, given by $e^{\pm i(r+2)\theta} y^{2r+2} e^{\pm isy^2}$. This can be alternately verified using the definition of $\eta^{\mp}$.

Definition 4. For $k \in \{0\} \cup \mathbb{N}$, define

$$I'_k(n,r,s) = \left\{ f \in I'(n,r,s) \mid \left(\partial_x^j f\right)(0,t) = 0 \text{ for } j = 0,1,\ldots,k \text{ and any } t \in \mathbb{R} \right\}$$

and write $I'_{-2}(n,r,s) = I'_{-1}(n,r,s) = I'(n,r,s)$. Similarly, define

$$I''_k(n,r,s) = \left\{ F \in I''(n,r,s) \mid \left(\partial_y^j F\right)(0,\theta) = 0 \text{ for } j = 0,1,\ldots,k \text{ and any } \theta \in \mathbb{R} \right\}$$

and write $I''_{-2}(n,r,s) = I''_{-1}(n,r,s) = I''(n,r,s)$. It is straightforward to check from Eq. (3.2) that if f corresponds to F, then $f \in I'_k(n,r,s)$ if and only if $F \in I''_k(n,r,s)$.

Note the actions given in Theorems 2 and 3 imply $I_k'(n,r,s)$ and $I_k''(n,r,s)$ are invariant under $SL(2,\mathbb{R})$.

Theorem 7. *Fix $s \neq 0$. The action of $H_3(\mathbb{R})$ preserves $\ker \nabla_{n,r,s}$ if and only if $r = -\frac{1}{2}$. The action of $SL(2,\mathbb{R})$ is given as follows.*

(1) *For $r \notin \{-\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots\}$, $\ker \nabla_{n,r,s} = \ker \nabla^+_{n,r,s}$ and is infinitesimally isomorphic to $\mathcal{S}^{n,r}$.*
(2) *For $r \in \{-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots\}$, $\ker \nabla_{n,r,s} = \ker \nabla^+_{n,r,s} \oplus \ker \nabla^-_{n,r,s}$.*
   (a) *The subrepresentation $\ker \nabla^+_{n,r,s}$ is infinitesimally isomorphic to $\mathcal{S}^{n,r}$.*
   (b) *The subrepresentation $\ker \nabla^-_{n,r,s} = \ker \nabla_{n,r,s} \cap I_{2r}'(n,r,s)$ is infinitesimally isomorphic to $\mathcal{S}^{n+1,-r-2}$.*
(3) *For $r \in \{0, 1, 2, \ldots\}$, $\ker \nabla_{n,r,s} = \ker \nabla^+_{n,r,s}$.*
   (a) *If $r \equiv n+1 \mod(2)$, then $\ker \nabla_{n,r,s} \subseteq I_{2r}'(n,r,s)$ is infinitesimally isomorphic to $\mathcal{S}^{n,-r-2}$.*
   (b) *If $r \equiv n$:*
      (i) *The subrepresentation $\ker \nabla_{n,r,s} \cap I_{2r}'(n,r,s)$ is infinitesimally isomorphic to $\mathcal{S}^{r,-r-2}$.*
      (ii) *The quotient $\ker \nabla_{n,r,s} / \left[\ker \nabla_{n,r,s} \cap I_{2r}'(n,r,s)\right]$ is infinitesimally isomorphic to V_r.*

Proof. For part (1), Lemma 2 says h_m exists if and only if $m \equiv n \mod (2)$ and can is uniquely determined by the normalization $h_m(0) = 1$. Now $\eta^\pm F_m$ is a multiple of $F_{m\pm 2}$ by $SL(2,\mathbb{R})$ theory. Since $F_m(y,\theta) = e^{-im\theta} h_m(y)$ and $\eta^\pm = \frac{1}{2} e^{\mp 2i\theta} \left[-y\partial_y \mp i\partial_\theta + (r \mp 2isy^2)\right]$,

$$\left(\eta^\pm F_m\right)(0,0) = \frac{1}{2}\,(\mp m + r) h_m(0) \tag{5.8}$$

so that $\left(\eta^\pm F_m\right) = \frac{1}{2}(r \mp m) F_{m\pm 2}$. As $\kappa F_m = m F_m$ (Eq. (3.3)), comparison to Eq. (4.1) and replacing F_m by $(-1)^{\lfloor \frac{m}{2} \rfloor} F_m$ finishes the argument.

For part (2), the argument showing $\ker \nabla^+_{n,r,s}$ is infinitesimally isomorphic to $\mathcal{S}^{n,r}$ is exactly the same as in part (1). It covers the case $m \equiv n$. When $m \equiv n+1$, the argument showing $\ker \nabla^-_{n,r,s}$ is infinitesimally isomorphic to $\mathcal{S}^{n+1,-r-2}$ is also similar, except that h_m is now normalized by the condition $h_m^{(2r+2)}(0) = 1$ with $h_m^{(j)}(0) = 0$ for all $0 \leqslant j < 2r+2$. Thus instead of using Eq. (5.8), substitute $k = 2r+2$ into the relation

$$(\partial_y^k \eta^\pm F_m)(0,0) = \frac{1}{2}\left[(-k \mp m + r)\, h_m^{(k)}(0) \mp 2isk(k-1) h_m^{(k-2)}(0)\right]$$

which is established by induction. In particular, this shows $\left(\eta^\pm F_m\right) = \frac{1}{2}(-r - 2 \mp m) F_{m\pm 2}$ and similarly finishes the argument that $\ker \nabla^-_{n,r,s}$ is infinitesimally isomorphic to $\mathcal{S}^{n+1,-r-2}$. The statement $\ker \nabla^-_{n,r,s} = \ker \nabla_{n,r,s} \cap I_{2r+1}'(n,r,s)$ follows from the construction of F_m in the proof of Lemma 2. In fact by the parity of h_m, it is equivalent to write $\ker \nabla^-_{n,r,s} = \ker \nabla_{n,r,s} \cap I'_{2r}(n,r,s)$.

For part (3), $m \equiv n$ and the same argument as in the second half of part (2) above shows $\mathrm{sp}_{\mathbb{C}}\{F_m\}$ is isomorphic to $\mathcal{S}^{n,-r-2}$. Thus (3a) and the first statement of (3b) follow. For the rest of (3b) follows from the proof of Lemma 2 by use of the intertwining map T_0.

The statement on $H_3(\mathbb{R})$ follows from Theorem 4 and simple examples as in the proofs of Theorems 4 and 6. □

6. Norms and intertwining maps

Lemma 4. *For $k \in \mathbb{N}$,*

$$\|f\|_k \equiv \left[\int_{-\infty}^{\infty} \left|\left(\partial_x^k f\right)(0,t)\right|^2 (1+t^2)^{-\mathrm{Re}\, r-1+k} dt\right]^{\frac{1}{2}}$$

is a well-defined seminorm on $I'_{k-2}(n,r,s)$. If $F \in I''_{k-2}(n,r,s)$ is the function corresponding to f, then

$$\|f\|_k^2 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left|\left(\partial_y^k F\right)(0,\theta)\right|^2 d\theta. \tag{6.1}$$

Moreover, the linear transformation

$$T_k(f) \equiv \left(\partial_x^k f\right)(0,\cdot)$$

maps $I'_{k-2}(n,r,s)$ to $\mathcal{P}^{n+k,-r-1+k}$, intertwines the $SL(2,\mathbb{R})$ action, and satisfies $\|f\|_k = \|T_k f\|_{\mathcal{P}^{n+k,-r-1+k}}$.

Proof. To check Eq. (6.1), use Eq. (3.2), a simple change of variables, and the product rule. This shows that the seminorm is well defined. The rest follows easily from the definitions. □

We use the notation $\ker T_k = \left\{f \in I'(n,r,s) \mid \left(\partial_x^k f\right)(0,t) = 0 \text{ for all } t \in \mathbb{R}\right\}$.

Theorem 8. *Fix $s \neq 0$.*

(1) *For $r \notin \{-\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots\}$, the seminorm $\|\cdot\|_0$ restricted to $\ker \nabla_{n,r,s} = \ker \nabla^+_{n,r,s}$ is a norm. With respect to this topology, the isometry $T_0 : \ker \nabla_{n,r,s} \to \mathcal{P}^{n,-r-1}$ completes to an $SL(2,\mathbb{R})$ intertwining isomorphism.*
(2) *Let $r \in \{-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots\}$. Then $\ker \nabla_{n,r,s} = \ker \nabla^+_{n,r,s} \oplus \ker \nabla^-_{n,r,s}$ with $\ker \nabla^-_{n,r,s} = \ker \nabla_{n,r,s} \cap I'_{2r}(n,r,s)$.*

(a) *The seminorm* $\|\cdot\|_0$ *restricted to* $\ker \nabla^+_{n,r,s}$ *is a norm. With respect to this topology, the isometry* $T_0 : \ker \nabla^+_{n,r,s} \to \mathcal{P}^{n,-r-1}$ *completes to an* $SL(2,\mathbb{R})$ *intertwining isomorphism.*

(b) *The seminorm* $\|\cdot\|_{2r+2}$ *restricted to* $\ker \nabla^-_{n,r,s}$ *is a norm. With respect to this topology, the isometry* $T_{2r+2} : \ker \nabla^-_{n,r,s} \to \mathcal{P}^{n+1,r+1}$ *completes to an* $SL(2,\mathbb{R})$ *intertwining isomorphism.*

(3) *For* $r \in \{0, 1, 2, \ldots\}$, $\ker \nabla_{n,r,s} = \ker \nabla^+_{n,r,s}$.

(a) *If* $r \equiv n+1 \mod(2)$, *then* $\ker \nabla_{n,r,s} \subseteq I'_{2r}(n,r,s)$ *and the seminorm* $\|\cdot\|_{2r+2}$ *restricted to* $\ker \nabla_{n,r,s}$ *is a norm. With respect to this topology, the isometry* $T_{2r+2} : \ker \nabla_{n,r,s} \to \mathcal{P}^{n,r+1}$ *completes to an intertwining isomorphism.*

(b) *If* $r \equiv n$:

(i) *The seminorm* $\|\cdot\|_{2r+2}$ *restricted to* $\ker \nabla_{n,r,s} \cap I'_{2r}(n,r,s)$ *is a norm. With respect to this topology, the isometry* $T_{2r+2} : \ker \nabla_{n,r,s} \cap I'_{2r}(n,r,s) \to \mathcal{P}^{r,r+1}$ *completes to an intertwining isomorphism.*

(ii) *The map* $T_0 : \ker \nabla_{n,r,s} \to V_r \subseteq \mathcal{P}^{r,-r-1}$ *is an intertwining map with kernel* $\ker \nabla_{n,r,s} \cap I'_{2r}(n,r,s)$. *The induced seminorm* $\|\cdot\|_0$ *on* $\ker \nabla_{n,r,s} / \left[\ker \nabla_{n,r,s} \cap I'_{2r}(n,r,s)\right]$ *is a norm and the induced isometry* $T_0 : \ker \nabla_{n,r,s} / \left[\ker \nabla_{n,r,s} \cap I'_{2r}(n,r,s)\right] \to V_r \subseteq \mathcal{P}^{r,-r-1}$ *is an intertwining isomorphism.*

Proof. Given the work already done, the most difficult part of the Theorem is to show the restricted seminorms are norms (though by Eq. (5.2) this is trivial on the K-finite vectors). Recall any $F \in \ker \nabla_{n,r,s}$ may be written as $F(y,\theta) = \sum_{m\in\mathbb{Z}} c_m e^{-im\theta} h_m(y)$ for some $c_m \in \mathbb{C}$ with uniform convergence on compact sets. By integrating, we see $c_m h_m(y) = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} F(y,\theta) e^{im\theta}\, d\theta$. In case (1) if $F(0,\theta) = 0$, then $c_m = 0$ for all m and so $F(y,\theta) = 0$. Since $\|f\|_0 = \|T_0 f\|_{\mathcal{P}^{n,-r-1}}$, we see $\ker \nabla_{n,r,s} \cap \ker T_0 = \{0\}$ so that the seminorm $\|\cdot\|_0$ restricted to $\ker \nabla_{n,r,s}$ is a norm. Possibly taking derivatives in y first, the other cases are handled similarly. The rest of the statements of the Theorem follows from Theorem 7 and Lemma 4. $\square$

7. Concluding remarks

We remark that the class of operators studied in the paper contain many interesting examples from physics. We have already noted the cases of the Heat and Schrödinger equations appear for the parameters $r = -\frac{1}{2}$ and $s = -\frac{1}{4}, -\frac{i}{4}$, respectively. Many other examples exist. For instance, the case of $r = s = -\frac{1}{2}$ realizes the Fokker–Planck equation

$$\left(\partial_x^2 + x\partial_x - \partial_t + 1\right) f = 0$$

after the substitution $f(x,t) \mapsto t^{\frac{1}{2}} f(t^{-\frac{1}{2}}x, \frac{1}{2}\ln t)$.

More generally, it is possible to study operators of the form

$$\begin{aligned}\nabla_{n,r,s,\alpha} &= \nabla_{n,r,s} + \alpha x^{-2} \\ &= \partial_x^2 - (1+2r)x^{-1}\partial_x + 4s\partial_t + \alpha x^{-2}\end{aligned}$$

for $\alpha \in \mathbb{C}$. As $\Omega = \frac{1}{2}x^2\nabla_{r,r,s,\alpha} + \frac{1}{2}[r(r+2)-\alpha]$, the $\ker \nabla_{n,r,s,\alpha} \subseteq I'(n,r,s)$ is still $SL(2,\mathbb{R})$-invariant. When $\alpha \neq 0$ and $s \neq 0$, it is straightforward to check (with techniques similar to those of the proof of Lemma 2) that smooth solutions on the K-finite level occur only for $r = \frac{k}{2} - 1 + \frac{\alpha}{2k}$ where $k \in \mathbb{N}$. For such an r, similar techniques to those used in this paper show that $\ker \nabla_{n,\frac{k}{2}-1+\frac{\alpha}{2k},s,\alpha} = \ker \nabla^{(\pm)^k}_{n,\frac{k}{2}-1+\frac{\alpha}{2k},s,\alpha} \cong \mathcal{P}^{n+k,\frac{k}{2}-\frac{\alpha}{2k}}$ via the map $f \mapsto (\partial^k f)(0,\cdot)$ when $\frac{\alpha}{k} \notin \mathbb{Z}$. Note the differential operator in this case is

$$\nabla_{n,\frac{k^2-2k+\alpha}{2k},s,\alpha} = \partial_x^2 - \left(k - 1 + \frac{\alpha}{k}\right) x^{-1}\partial_x + 4s\partial_t + \alpha x^{-2}.$$

When $\frac{\alpha}{k} \in \mathbb{Z}$, in addition to the $\mathcal{P}^{n+k,\frac{k}{2}-\frac{\alpha}{2k}}$ component discussed above, there can be a quotient of $\ker \nabla_{n,\frac{k}{2}-1+\frac{\alpha}{2k},s,\alpha}$ that maps into $\mathcal{P}^{n+k,-\frac{k}{2}+\frac{\alpha}{2k}}$ via $f \mapsto (\partial^{\frac{\alpha}{k}} f)(0,\cdot)$. For instance, when $j = \frac{k}{2} - \frac{\alpha}{2k} - 1 \in \mathbb{N}$ and $n \equiv j - \frac{\alpha}{k}$, V_j is realized.

Additionally, if one is willing to throw out the t-axis, it is possible to realize the representations studied in this paper as induced representations from lower *unipotent* matrices up to $SL(2,\mathbb{R})$. This method involves similar ideas to this paper and requires a careful choice of an initial section to get a proper factor of automorphy. In this way one may deal with singular solutions to the differential equation. However, other topological problems arise.

Finally, it seems likely that the proper generalization of these results will involve the theory of dual pairs.

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